Quadratures with Remainders of Minimum Norm. II

By R. E. Barnhill* and J. A. Wixom

1. Introduction. Let the quadrature remainder with n base points be given by

$$R_n(f) = \int_{-1}^{1} f - \sum_{k=1}^{n} A_k f(z_k) \tag{1}$$

where f is in the Hilbert space $L^2(E_\rho)$. $L^2(E_\rho)$ is $\{f(z): f$ is analytic inside the ellipse E_ρ and $\iint_{E_\rho} |f(z)|^2\, dx\, dy$ exists$\}$, where E_ρ is the ellipse with foci at ± 1, semimajor axis a, semiminor axis $b = (a^2 - 1)^{1/2}$ and $\rho = (a + b)^2$, and the double integral is taken over the region inside the ellipse. For additional information on the space $L^2(E_\rho)$ the reader is referred to Davis [5]. For fixed n, R_n is a bounded linear functional on $L^2(E_\rho)$. The problem is to minimize $||R_n|| = \sup\,(|R_n(f)|/||f||)$ by an appropriate choice of the A_k and z_k in Eq. (1). In [2] the problem of minimizing $||R_n||$ with respect to the A_k was solved, and this paper extends those results to the case of variable base points z_k.

The idea of minimizing the norm of the remainder has appeared in several papers. For the Hardy space H_2, Yanagihara [9] posed it for 2-, 3- and 4-point quadrature rules and obtained explicit solutions for the weights and points. The first author rediscovered some of Yanagihara's results and also solved the minimization problem for the space $L^2(E_\rho)$ in his doctoral dissertation [10]. Valentin extended some of Yanagihara's results for the space H_2 and he also considered the space $L^2(R)$ (R being the unit disc) in his doctoral dissertation [8]. For the space H_2, Wilf [11] has also considered this problem. In the latter three papers the cases solved were done numerically. The problem is also mentioned in Davis [12].

2. Minimization of the Norm of the Remainder. For an arbitrary normed linear space X, it is difficult to find a representation of $||R_n||$ that can be computed. However, since $L^2(E_\rho)$ is a Hilbert space, the Riesz representation theorem for Hilbert space can be used to find a computable representation of $||R_n||$. This idea was first applied to quadratures by Davis [3]. Specifically, if $\{P_m(z)\}_{m=0}^{\infty}$ is a complete orthonormal sequence in $L^2(E_\rho)$, then

$$||R_n||^2 = \sum_{m=0}^{\infty} |R_n(P_m)|^2 = \sum_{m=0}^{\infty} \left| \int_{-1}^{1} P_m(z)dz - \sum_{k=1}^{n} A_k P_m(z_k) \right|^2 .$$

For the space $L^2(E_\rho)$, the complete orthonormal sequence can be defined as follows: $P_m(z) = 2(m+1)^{1/2}[\pi(\rho^{m+1} - \rho^{-m-1})]^{-1/2} U_m(z)$, where $U_m(z) = (1 - z^2)^{-1/2} \times \sin[(m+1) \arc\cos z]$, $m = 0, 1, \cdots$. Then

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$$||R_n||^2 = \sum_{m=0}^{\infty} \alpha(m, \rho) \left| \beta(m) - \sum_{k=1}^{n} A_k U_m(z_k) \right|^2, \tag{2}$$

where $\alpha(m, \rho) = 4(m + 1)/[\pi(\rho^{m+1} - \rho^{-m-1})]$; $\beta(m) = [1 + (-1)^m]/(m + 1)$. $||R_n||$ is a continuous function of the A_k and z_k. The z_k are assumed real for the cases considered and this forces the A_k to be real also. If we consider the A_k and z_k as belonging to a compact region in Euclidean $2n$-space, say, $|A_k| \leqq 1$, $|z_k| \leqq 1$, $k = 1, \cdots, n$, then $||R_n||$ has a minimum in the region.

In order to calculate this minimum, we set $\partial||R_n||^2/\partial A_k = 0$, $\partial||R_n||^2/\partial z_k = 0$, $k = 1, \cdots, n$ and solve the resulting nonlinear system of $2n$ equations in $2n$ variables. The equations to be solved are the following:

$$\begin{aligned} &\sum_{m=0}^{\infty} 2\alpha(m, \rho) \left(\beta(m) - \sum_{k=1}^{n} A_k U_m(z_k) \right) (-U_m(z_j)) = 0, \qquad j = 1, \cdots, n, \\ &\sum_{m=0}^{\infty} 2\alpha(m, \rho) \left(-A_j \left(\beta(m) - \sum_{k=1}^{n} A_k U_m(z_k) \right) U'_m(z_j) \right) = 0, \qquad j = 1, \cdots, n. \end{aligned} \tag{3}$$

Newton's method is used to solve the system of Eqs. (3). The initial approximations to the z_k are the Gaussian base points corresponding to the same value of n. The initial approximations to the A_k are the A_k^* which minimize $||R_n||$, with the z_k fixed as the Gaussian points. The A_k^* are given in reference [2].

3. Examples and Use of the Tables. Tables of the minimum $||R_n||$ and the minimizing A_k and z_k, for various values of n and ellipses E_ρ, are given in Section 4.

In this section, we consider the use of the minimum $||R_n||$ to estimate the quadrature error $|R_n(f)|$ and we also compare $|R_n(f)|$, using the minimizing A_k and z_k, with $|R_n(f)|$ for known quadrature rules. The upper bound used is $|R_n(f)| \leqq ||R_n|| \cdot ||f||$, where $||f||^2 = \int\int_{E_\rho} |f(z)|^2 \, dx \, dy$. An upper bound must usually be used for $||f||$. One such bound is $M(\pi ab)^{1/2}$, where M is the supremum of $|f|$ inside the ellipse E_ρ. If f is analytic on the ellipse, then M is the maximum of $|f|$ and it occurs on the ellipse.

Example 1. f is analytic on the ellipse E_ρ and $M = \sup_{z \epsilon E_\rho} |f(z)| = e^{a^2}$, for $f(z) = e^{z^2}$. Since $b = (a^2 - 1)^{1/2}$, we have $|R_n(e^{z^2})| \leqq ||R_n|| \cdot ||e^{z^2}|| \leqq ||R_n|| e^{a^2}[\pi a(a^2 - 1)^{1/2}]^{1/2}$. This gives an error bound for $f(z) = e^{z^2}$ as a function of n and a. For each n we select the value of a from the tables which minimizes this expression. The minimizing values are shown in the table below.

n	*Minimizing value of a*	$\lvert\lvert R_n \rvert\rvert e^{a^2}[\pi a(a^2 - 1)^{1/2}]^{1/2}$
2	1.50	1.26776
3	2.0	0.15599
4	2.0	0.01290

Example 2. We have $M = a(e^{4b} + e^{-4b})/2$ and $|R_n(z \cos z \sin z)| \leqq ||R_n|| \cdot M(\pi ab)^{1/2}$, for $f(z) = z \cos z \sin z$.

The minimizing values are shown in the table below.

n	*Minimizing value of a*	$\lvert\lvert R_n \rvert\rvert \cdot M(\pi ab)^{1/2}$
2	1.03	2.25136
3	1.03	1.78641
4	1.40	0.87444

The following table contains comparisons, for specific functions, of minimum norm (MN) quadratures with various known quadratures. Composite rules are used on the functions $1/(1 + z^2)$ and $z \sin z \cos z$ with step-lengths as indicated. The numbers in parentheses indicate the appropriate power of 10. For each function the same number of base points was used for MN quadratures as for the known quadratures. The Tchebycheff quadratures are the quadratures with equal weights that have the highest polynomial precision [7].

			Error Using MN Quadratures		*Error Using Known Quadratures*	
Function	*Interval of integration*	*Number of base pts.*	a	*Error*	*Quadrature*	*Error*
$\lvert z\rvert$	$[-1, 1]$	3	1.30	0.13896	Gauss	0.13934
			1.40	0.13844	Newton-Cotes	0.33333
			1.50	0.13851	Tchebycheff	0.05719
$\lvert z\rvert$	$[-1, 1]$	4	1.10	−0.02871	Gauss	−0.04254
			1.15	−0.03862	Newton-Cotes	0.0
			1.75	−0.04246	Tchebycheff	0.01775
$1/(1 + z^2)$	$[-4, 4]$	3	1.10	−1.09576	Gauss	−1.32321
			1.15	−1.22532	Newton-Cotes	−2.83856
			2.50	−1.32233	Tchebycheff	−0.60762
$1/(1 + z^2)$*	$[-4, 4]$	4	1.50	−0.73376(−06)	Gauss	−0.98048(−05)
			1.75	−0.90680(−05)	Newton-Cotes	0.14745(−02)
			2.00	−0.95592(−05)	Tchebycheff	0.18212(−03)
$(9 + 2z)^{-1/2}$	$[-4, 4]$	3	1.50	0.04187	Gauss	0.04061
			1.75	0.04101	Newton-Cotes	−0.31139
			2.00	0.04079	Tchebycheff	0.07923
$z \sin z \cos z$	$[-1, 1]$	3	1.40	0.00475	Gauss	0.00517
			1.50	0.00478	Newton-Cotes	0.13230
			1.75	0.00494	Tchebycheff	−0.03024
$z \sin z \cos z$*	$[-1, 1]$	4	1.75	−0.18523(−06)	Gauss	−0.24259(−06)
			2.00	−0.22418(−06)	Newton-Cotes	0.14549(−02)
			2.50	−0.23270(−06)	Tchebycheff	0.26884(−04)

* Composite rule with step-length 1.0.

TABLE 1

$N = 2$

a	*Base Points*	*Weights*	$\|\|R_2\|\|$
1.03	0.5306967015	0.5242087319	1.7385340982
1.05	0.5389972688	0.6575665167	1.2883434873
1.10	0.5519030316	0.8369649737	0.7293161604
1.15	0.5592979275	0.9152367390	0.4623701537
1.20	0.5639700051	0.9527037191	0.3127386455
1.25	0.5671105812	0.9720726463	0.2213011434
1.30	0.5693184230	0.9827374321	0.1620129721
1.40	0.5721257073	0.9926623836	0.0936211470
1.50	0.5737590630	0.9965263751	0.0582140241
1.75	0.5757005520	0.9992657692	0.0214811009
2.00	0.5764713404	0.9997914963	0.0099094274
2.50	0.5770260520	0.9999716218	0.0028420266
Gauss	0.5773502692	1.0	

TABLE 2

$N = 3$

a	*Base Points*	*Weights*	$\|\|R_3\|\|$
1.03	0.7434834252	0.4015017486	1.3800704854
	0.0	0.6003729582	
1.05	0.7518233122	0.4749670772	0.8937754839
	0.0	0.7203543980	
1.10	0.7623021863	0.5384360267	0.3828139543
	0.0	0.8322752623	
1.15	0.7669501499	0.5530018003	0.1960803668
	0.0	0.8630079016	
1.20	0.7694119638	0.5568194848	0.1115324621
	0.0	0.8741094499	
1.25	0.7708708741	0.5577469582	0.0680827745
	0.0	0.8791198738	
1.30	0.7718054048	0.5578103560	0.0437555480
	0.0	0.8818136908	
1.40	0.7728879061	0.5573648268	0.0201919851
	0.0	0.8845753232	
1.50	0.7734643431	0.5569025309	0.0103573945
	0.0	0.8859711882	
1.75	0.7740993485	0.5562167388	0.0026201244
	0.0	0.8875450457	
2.00	0.7743365086	0.5559146211	0.0008661110
	0.0	0.8881675221	
2.50	0.7745019720	0.5556895392	0.0001506814
	0.0	0.8886207597	
Gauss	0.7745966692	0.5555555556	
	0.0	0.8888888889	

TABLE 3
$N = 4$

a	*Base Points*	*Weights*	$\|\|R_4\|\|$
1.03	0.8434055237 0.3283257294	0.3019737608 0.5308958137	1.0316186099
1.05	0.8495395476 0.3319553911	0.3342347346 0.5977818841	0.5717864022
1.10	0.8557804260 0.3357683847	0.3503185979 0.6390052212	0.1845142780
1.15	0.8580390968 0.3372551809	0.3512050953 0.6463753888	0.0770467932
1.20	0.8591144634 0.3380354752	0.3506375343 0.6486767179	0.0371216097
1.25	0.8597141460 0.3385155033	0.3500424633 0.6497312377	0.0196398593
1.30	0.8600844267 0.3388388676	0.3495766937 0.6503397858	0.0111137456
1.40	0.8605008925 0.3392399970	0.3489647267 0.6510207626	0.0041087299
1.50	0.8607177992 0.3394709812	0.3486096510 0.6513871622	0.0017410793
1.75	0.8609535029 0.3397457245	0.3481958730 0.6518039877	0.0002973320
2.00	0.8610408334 0.3398553575	0.3480351680 0.6519648209	0.0000716323
2.50	0.8611015909 0.3399345844	0.3479209825 0.6520790173	0.0000075609
Gauss	0.8611363116 0.3399810436	0.3478548451 0.6521451549	

4. Tables. Tables 1, 2, and 3 list the values of the quadrature weights A_k and base points z_k, and the corresponding values obtained for $||R_n||$ from Eq. (2), for $n = 2, 3, 4$, respectively. The minimizing values of the z_k are symmetric; hence, only the nonnegative ones are listed. The weights obtained for symmetric base points are equal and so only those weights corresponding to nonnegative base points are listed.

5. Conclusions. For the numerous functions tested minimum norm quadratures were, overall, comparable in accuracy to Gaussian quadratures and better than Newton-Cotes and Tchebycheff quadratures. It is generally the case that composite rules must be used to achieve sufficient accuracy in a practical problem and the quadratures of the function $z \sin z \cos z$ given in Section 3 illustrate the use and accuracy of a composite minimum norm quadrature. It might be noted that the MN rules do not integrate constants exactly and so those theorems requiring the sum of the weights to equal the length of the interval do not apply.

The MN quadratures have interesting asymptotic properties, both as $\rho \to \infty$ and as $n \to \infty$. From Tables 1, 2 and 3 it can be seen numerically that the weights and base points of the MN quadratures seem to approach the weights and base points of the Gaussian quadratures with the same number of points. Valentin [8] has proved a similar result and his proof can be altered to prove the above conjecture, the details of which will appear in a future paper.

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[M,X] - R. E. Barnhill and J. A. Wixom, Tables Related to Quadratures with Remainders of Minimum Norm I, ms. of 22 typewritten pages deposited in the UMT File.

These tables contain the weights w_k for a family of quadrature formulas of the following type:

$$\int_{-1}^{+1} f(x)\,dx = \sum_{k=1}^{n} w_k f(x_k) + R_n,$$

where R_n denotes the error associated with using the sum in place of the integral. Different groups of weights are tabulated, one for each of ten sets of abscissas x_1, x_2, ..., x_n. These sets of abscissas are identical to those used in the following rules: trapezoidal, Simpson, Weddle, and Gauss 2, 3, 4, 5, 7, 10, 16 point rules. A bound for the quadrature error of the form

$$|R_n| \le ||R_n|| \; ||f||$$

exists. The norm $||R_n||$ (c.f. reference [1]) is also tabulated. The norm $||f||$ is defined by

$$||f|| = \iint_{\varepsilon(a)} |f(z)|^2 \,dxdy$$

or by the same relation with f(z) replaced by f'(z), the first derivative of f(z), depending on the choice of tabulated weights; the double integral is taken over an ellipse in the complex plane with semimajor axis, a, and semiminor axis, $b = (a^2 - 1)^{1/2}$. Weights are tabulated for different values of a ranging from 1.0001 to 5.0. These weights have been determined for each value of a and each set of abscissas by the condition that the norm $||R_n||$ be minimized. It is therefore possible for these weights to yield a smaller quadrature error than that associated with the corresponding ''ordinary'' weights and same abscissas; comparison of the quadrature errors for some special cases is given in reference 1.

Eleven digit numbers are tabulated; the calculations were carried out in double precision (16 digits). The results of $||R_n||$, using the standard weights, agreed with the results obtained by Lo, Lee and Sun,[2] which gives an external check on the computations. An explanation of the headings - NO PRECISION - and -PRECISION FOR CONSTANTS - can be found in reference [1].

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DOUBLE INTEGRAL NORM - NO PRECISION - QUADRATURE WEIGHTS

1	TRAPEZOID	SIMPSON	GAUSS 2 PT	GAUSS 3 PT	GAUSS 4 PT
001	0.10000000000D 01	0.16211119200D-03	0.10000000000D 01	0.22776211590D-01	0.18307748282D-01
		0.36012352548D-01		0.36012352548D-01	0.33867179135D-01
070		0.11334757767D-01		0.19040145434D 00	0.15276614640D 00
		0.30112798194D 00		0.30110936182D 00	0.28296063775D 00
090		0.14568415074D-01		0.21574596292D 00	0.17267489403D 00
		0.34139026219D 00		0.34130573527D 00	0.32028560251D 00
100		0.16184435464D-01		0.22730183621D 00	0.18162267493D 00
		0.35982698160D 00		0.35967542070D 00	0.33716722520D 00
200		0.32313053773D-01		0.31747125765D 00	0.24668655036D 00
		0.50842457032D 00		0.50467882192D 00	0.46234598488D 00
300		0.48361078436D-01		0.37908743908D 00	0.28412793600D 00
		0.62190138805D 00		0.60611290281D 00	0.53597916953D 00
400		0.64255262545D-01		0.42302735792D 00	0.30645649419D 00
		0.71651498132D 00		0.67959370778D 00	0.57977472947D 00
500		0.79884291091D-01		0.45483480248D 00	0.32016274722D 00
		0.79816294220D 00		0.73313175942D 00	0.60613629141D 00
000		0.15020278474D 00		0.52537861849D 00	0.34260959376D 00
		0.10802156271D 01		0.84982939513D 00	0.64587820577D 00
500		0.20282435380D 00		0.54419488775D 00	0.34640351403D 00
		0.12273873780D 01		0.87778922456D 00	0.65100384877D 00
000		0.23917244250D 00		0.55052716720D 00	0.34735271626D 00
		0.13006864146D 01		0.88562617559D 00	0.65191536359D 00
500		0.26383662609D 00		0.55305580149D 00	0.34765350020D 00
		0.13357453855D 01		0.88805225130D 00	0.65210555948D 00
000		0.28075945513D 00		0.55420054206D 00	0.34776511657D 00
		[illegible]		[illegible]	[illegible]

a	SIMPSON	GAUSS 3 PT	GAUSS 4 PT
4000	0.301164361750 00	0.55507595464D 00	0.34783258104D 00
	0.13595789490D 01	0.88909969008D 00	0.65215202334D 00
5000	0.31218420402D 00	0.55535361671D 00	0.34784804875D 00
	0.13573247426D 01	0.88905236490D 00	0.65214857350D 00
7500	0.32420798817D 00	0.55551891264D 00	0.34785423425D 00
	0.13476570582D 01	0.88893916235D 00	0.65214561823D 00
0000	0.32860624413D 00	0.55554580970D 00	0.34785475555D 00
	0.13416669524D 01	0.88890491738D 00	0.65214523265D 00
5000	0.33162658536D 00	0.55555429277D 00	0.34785484074D 00
	0.133659384230 01	0.88889124123D 00	0.65214515904D 00
0000	0.33255885909D 00	0.55555529603D 00	0.34785484472D 00
	0.13348503035D 01	0.88888939143D 00	0.65214515527D 00
0000	0.333101591530 00	0.55555553234D 00	0.34785484511D 00
	0.133379392810 01	0.88888893486D 00	0.65214515489D 00
0000	0.33324069900D 00	0.55555555184D 00	0.34785484545D 00
	0.13335181393D 01	0.88888889628D 00	0.65214515455D 00

OOUBLE INTEGRAL NORM - NO PRECISION - QUADRATURE WEIGHTS

α	GAUSS 5 PT	WEODLE	GAUSS 7 PT	GAUSS 10 PT	GAUSS 16 PT
1.0001	0.15229354662D-01	0.16211119200D-03	0.11342128638D-01	0.817298525850-02	0.52293335912D-02
	0.30345633215D-01	0.26842022775D-01	0.24161349987D-01	0.18066050676D-01	0.11822698178D-01
	0.36012352548D-01	0.339527715900-01	0.329131962660-01	0.26424396834D-01	0.18030742222D-01
		0.36012352548D-01	0.36012352548D-01	0.32454471824D-01	0.23597413121D-01
				0.35611035306D-01	0.28315625425D-01
					0.32012948273D-01
					0.34554962844D-01
					0.35849435304D-01
1.0070	0.12630370087D 00	0.11332248848D-01	0.90784829021D-01	0.58106823458D-01	0.26679696484D-01
	0.25266404247D 00	0.219684528470 00	0.195864647140 00	0.13136734048D 00	0.61549233347D-01
	0.29987062487D 00	0.266400951960 00	0.266929521900 00	0.19229778353D 00	0.93953302756D-01
		0.276758035590 00	0.292090020330 00	0.23628133506D 00	0.12307551451D 00
				0.25929501305D 00	0.14772130507D 00
					0.16703632220D 00
					0.18031328235D 00
					0.18707416844D 00
1.0090	0.14191340423D 00	0.14556524187D-01	0.997094218460-01	0.61187036095D-01	0.26931211276D-01
	0.28471766522D 00	0.244922233610 00	0.216051546850 00	0.13849231641D 00	0.61997087294D-01
	0.33793980496D 00	0.288740968910 00	0.294489721090 00	0.20272174960D 00	0.94662288178D-01
		0.296841483270 00	0.32226627772D 00	0.24912525221D 00	0.12400908519D 00
				0.27339783317D 00	0.14884119566D 00
					0.16830477588D 00
					0.18168280636D 00
					0.18849541956D 00
1.0100	0.14873529583D 00	0.16162701608D-01	0.103274863880 00	0.62229657700D-01	0.26995974383D-01
	0.29885424042D 00	0.25578188921D 00	0.224152286990 00	0.14084818379D 00	0.62097096565D-01
	0.35473422874D 00	0.29702718192D 00	0.305548196270 00	0.20616585823D 00	0.94827883521D-01
		0.303904334360 00	0.334377539870 00	0.25337343986D 00	0.12422569160D 00
				0.27806240752D 00	0.14910108733D 00
					0.16859932081D 00
					0.18200074443D 00
					0.18882540813D 00

a	GAUSS 5 PT	WEDDLE	GAUSS 7 PT	GAUSS 10 PT	GAUSS 16 PT
1.0200	0.19329403085D 00	0.31695075215D-01	0.12119857196D 00	0.65912142440D-01	0.27142059170D-01
	0.39280738766D 00	0.32867425275D 00	0.26434008275D 00	0.14848029924D 00	0.62256671478D-01
	0.46636205951D 00	0.32690859490D 00	0.36032436589D 00	0.21739621621D 00	0.95143054629D-01
		0.32965620257D 00	0.39443181539D 00	0.26725241831D 00	0.12461999272D 00
				0.29329491520D 00	0.14957952241D 00
					0.16914052859D 00
					0.18258498313D 00
					0.18943190737D 00
1.0300	0.21381861721D 00	0.45489891989D-01	0.12631171676D 00	0.66475729374D-01	0.27150993850D-01
	0.43609677948D 00	0.37102135224D 00	0.27484094275D 00	0.14932159079D 00	0.62255187016D-01
	0.51766807514D 00	0.32060922732D 00	0.37464748212D 00	0.21874075928D 00	0.95156219147D-01
		0.33873594233D 00	0.41019019983D 00	0.26889861458D 00	0.12462930600D 00
				0.29510187146D 00	0.14959436874D 00
					0.16915581465D 00
					0.18260207478D 00
					0.18944945921D 00
1.0400	0.22390262169D 00	0.56995759219D-01	0.12808784948D 00	0.66607211130D-01	0.27152157357D-01
	0.45662713263D 00	0.39981474826D 00	0.27801614073D 00	0.14944278612D 00	0.62254026047D-01
	0.54191898871D 00	0.30483209062D 00	0.37906327393D 00	0.21898989801D 00	0.95157941258D-01
		0.35136590119D 00	0.41505795514D 00	0.26918582103D 00	0.12462927790D 00
				0.29542174352D 00	0.14959564337D 00
					0.16915656886D 00
					0.18260322277D 00
					0.18945051885D 00
1.0500	0.22918259101D 00	0.66232013282D-01	0.12880400215D 00	0.66646696876D-01	0.27152380894D-01
	0.46679914907D 00	0.42045660462D 00	0.27909332287D 00	0.14945899202D 00	0.62253681691D-01
	0.55393287086D 00	0.28643878741D 00	0.38063477499D 00	0.21905224238D 00	0.95158333796D-01
		0.36758419801D 00	0.41678007654D 00	0.26924795523D 00	0.12462910007D 00
				0.29549448823D 00	0.14959587879D 00
					0.16915656935D 00
					0.18260336638D 00
					0.18945060667D 00
1.1000	0.23593149740D 00	0.89535318520D-01	0.12943950451D 00	0.66670620715D-01	0.27152458890D-01
	0.47791462274D 00	0.46796307514D 00	0.27972142582D 00	0.14945251430D 00	0.62253525361D-01
	0.56743066540D 00	0.20740009682D 00	0.38176964721D 00	0.21908513040D 00	0.95158509759D-01
		0.45404143032D 00	0.41794438583D 00	0.26926723907D 00	0.12462897319D 00
				0.29552382438D 00	0.14959598727D 00
					0.16915652060D 00
					0.18260341431D 00

α	GAUSS 5 PT	WEDDLE	GAUSS 7 PT	GAUSS 10 PT
1.1500	0.23670697670D 00	0.96234393040D-01	0.12947833894D 00	0.66671286395D-01
	0.47862646510D 00	0.48394245331D 00	0.27971312235D 00	0.14945147366D 00
	0.56859422977D 00	0.16084110062D 00	0.38181977299D 00	0.21908623057D 00
		0.51405492130D 00	0.41796404479D 00	0.26926680450D 00
				0.29552419042D 00
1.2000	0.23686112451D 00	0.98287571236D-01	0.12948352716D 00	0.66671336838D-01
	0.47866585124D 00	0.49198363582D 00	0.27970782784D 00	0.14945136719D 00
	0.56879554172D 00	0.13329493649D 00	0.38182720215D 00	0.21908634218D 00
		0.55172715006D 00	0.41796145140D 00	0.26926673390D 00
				0.29552421931D 00
1.2500	0.23690299726D 00	0.98889903738D-01	0.12948456740D 00	0.66671343008D-01
	0.47865300574D 00	0.49703071261D 00	0.27970620656D 00	0.14945135249D 00
	0.56885067624D 00	0.11580588174D 00	0.38182908276D 00	0.21908635854D 00
		0.57616481806D 00	0.41796008462D 00	0.26926672231D 00
				0.29552422360D 00
1.3000	0.23691694967D 00	0.98994428855D-01	0.12948483576D 00	0.66671344044D-01
	0.47864229251D 00	0.50055215499D 00	0.27970568968D 00	0.14945134985D 00
	0.56887084058D 00	0.10401310276D 00	0.38182968164D 00	0.21908636166D 00
		0.59273828889D 00	0.417959551[illegible] 00	0.26926671998D 00
				0.29552422447D 00
1.4000	0.23692462249D 00	0.98797691237D-01	0.12948494671D 00	0.66671344258D-01
	0.47863287190D 00	0.50512057795D 00	0.27970544187D 00	0.14945134928D 00
	0.56888382173D 00	0.89592873133D-01	0.38182998331D 00	0.21908636239D 00
		0.61295274134D 00	0.41795925464D 00	0.26926671938D 00
				0.29552422470D 00
1.5000	0.23692622672D 00	0.98533121785D-01	0.12948496227D 00	0.66671515947D-01
	0.47863008978D 00	0.50787130191D 00	0.27970540226D 00	0.14945083919D 00
	0.56888718883D 00	0.81480556873D-01	0.38183003539D 00	0.21908708223D 00
		0.62422452676D 00	0.41795920007D 00	0.26926608586D 00
				0.29552447677D 00
1.7500	0.23692682899D 00	0.98091741370D-01	0.12948497001D 00	
	0.47862881183D 00	0.51126750635D 00	0.27970537932D 00	
	0.56888871481D 00	0.72082081550D-01	0.38183007014D 00	
		0.63711709917D 00	0.41795916106D 00	

a	GAUSS 5 PT	WEDDLE	GAUSS 7 PT
2.0000	0.236926877010D 00	0.978813672240D-01	0.129484807850D 00
	0.478628691970D 00	0.512661678910D 00	0.279705887990D 00
	0.568888861910D 00	0.684176585400D-01	0.381829258900D 00
		0.642078571220D 00	0.417960090520D 00
2.5000	0.236926885180D 00	0.977870867560D-01	
	0.478628670130D 00	0.513268958220D 00	
	0.568888889390D 00	0.668401837930D-01	
		0.644207542040D 00	
3.0000	0.236926902500D 00		
	0.478628620890D 00		
	0.568888953240D 00		
4.0000	0.236927478780D 00		
	0.478626984560D 00		
	0.568891073330D 00		
5.0000	0.236899202560D 00		
	0.478707120050D 00		
	0.568787354770D 00		

DOUBLE INTEGRAL NORM - PRECISION FOR CONSTANTS - QUADRATURE WEIGHTS

α	TRAPEZOID	SIMPSON	GAUSS 2 PT	GAUSS 3 PT	GAUSS 4 PT
1.0001	0.10000000000D 01	0.36125947505D 00 0.12774810499D 01	0.10000000000D 01	0.58349842077D 00 0.83300315846D 00	0.37214909189D 00 0.62785090811D 00
1.0070		0.34868294527D 00 0.13026341095D 01		0.57238507026D 00 0.85522985948D 00	0.35883315114D 00 0.64116684886D 00
1.0090		0.34722120282D 00 0.13055575944D 01		0.57095804502D 00 0.85808390997D 00	0.35735593058D 00 0.64264406942D 00
1.0100		0.34658888888D 00 0.13068222222D 01		0.57033109771D 00 0.85933780459D 00	0.35673073422D 00 0.64326926578D 00
1.0200		0.34228951967D 00 0.13154209607D 01		0.56591324206D 00 0.86817351588D 00	0.35279750008D 00 0.64720249992D 00
1.0300		0.33985563522D 00 0.13202887296D 01		0.56328993906D 00 0.87342012189D 00	0.35090717201D 00 0.64909282799D 00
1.0400		0.33828477346D 00 0.13234304531D 01		0.56154414594D 00 0.87691170811D 00	0.34985858380D 00 0.65014141620D 00
1.0500		0.33719967219D 00 0.13256006556D 01		0.56030920369D 00 0.87938159262D 00	0.34922736339D 00 0.65077263661D 00
1.1000		0.33476544379D 00 0.13304691124D 01		0.55741546447D 00 0.88516907106D 00	0.34815569111D 00 0.65184430889D 00
1.1500		0.33399684654D 00 0.13320063069D 01		0.55644421539D 00 0.88711156923D 00	0.34794805870D 00 0.65205194130D 00
1.2000		0.33368120181D 00 0.13326375964D 01		0.55603048128D 00 0.88793903744D 00	0.34788981861D 00 0.65211018139D 00
1.2500		0.33353120643D 00 0.13329375871D 01		0.55582916456D 00 0.88834167089D 00	0.34786972826D 00 0.65213027174D 00
1.3000		0.33345272651D 00 0.13330945470D 01		0.55572211884D 00 0.88855576232D 00	0.34786178021D 00 0.65213821979D 00

α	SIMPSON	GAUSS 3 PT	GAUSS 4 PT
1.4000	0.333382671580 00	0.555625160760 00	0.347856671160 00
	0.133323465680 01	0.888749678480 00	0.652143328840 00
1.5000	0.333356340360 00	0.555588223580 00	0.347855423970 00
	0.133328731930 01	0.888823552840 00	0.652144576030 00
1.7500	0.333338113350 00	0.555562394170 00	0.347854895710 00
	0.133332377330 01	0.888875211660 00	0.652145104290 00
2.0000	0.333334682860 00	0.555557492220 00	0.347854853110 00
	0.133333063430 01	0.888885015560 00	0.652145146890 00
2.5000	0.333333513040 00	0.555555813300 00	0.347854840460 00
	0.133333297390 01	0.888888373400 00	0.652145159540 00
3.0000	0.333333368620 00	0.555555605700 00	0.347854840620 00
	0.133333326280 01	0.888888788590 00	0.652145159380 00
4.0000	0.333333336350 00	0.555555559810 00	0.347854844470 00
	0.133333332730 01	0.888888880370 00	0.652145155530 00
5.0000	0.333333333460 00	0.555555555640 00	0.347854844490 00
	0.133333333310 01	0.888888888720 00	0.652145155510 00

DOUBLE INTEGRAL NORM - PRECISION FOR CONSTANTS - QUADRATURE WEIGHTS

α	GAUSS 5 PT	WEDDLE	GAUSS 7 PT	GAUSS 10 PT	GAUSS 16 PT
1.0001	0.25555501361D 00	0.11275743784D 00	0.14040893377D 00	0.72236485637D-01	0.29173151185D-01
	0.46677859362D 00	0.38158540826D 00	0.27598917179D 00	0.14840008395D 00	0.62032844916D-01
	0.55533278555D 00	0.33740678757D 00	0.37716789395D 00	0.21781896419D 00	0.94968457426D-01
		0.33650073268D 00	0.41286800097D 00	0.26771743342D 00	0.12438131499D 00
				0.29382703281D 00	0.14930041244D 00
					0.16882290956D 00
					0.18224355249D 00
					0.18907735699D 00
1.0070	0.24333841480D 00	0.10991139937D 00	0.13159612920D 00	0.67094283724D-01	0.27173460236D-01
	0.47414301340D 00	0.39748807118D 00	0.27869819927D 00	0.14926623042D 00	0.62240461688D-01
	0.56503714360D 00	0.32324452068D 00	0.38114837305D 00	0.21904303470D 00	0.95161872244D-01
		0.33871201755D 00	0.41711459694D 00	0.26916935626D 00	0.12462572640D 00
				0.29542709489D 00	0.14959474774D 00
					0.16915421644D 00
					0.18260124865D 00
					0.18944826659D 00
1.0090	0.24219241083D 00	0.10998238941D 00	0.13103868787D 00	0.66935029399D-01	0.27162001221D-01
	0.47487123939D 00	0.40032646510D 00	0.27891969611D 00	0.14932201329D 00	0.62246519707D-01
	0.56587269957D 00	0.31931355523D 00	0.38136743009D 00	0.21906885071D 00	0.95160865270D-01
		0.34075518052D 00	0.41734837187D 00	0.26920655947D 00	0.12462718193D 00
				0.29546754712D 00	0.14959570465D 00
					0.16915549406D 00
					0.18260257931D 00
					0.18944965385D 00
1.0100	0.24172696517D 00	0.11002886599D 00	0.13083115490D 00	0.66883113409D-01	0.27159103129D-01
	0.47517046518D 00	0.40165143107D 00	0.27900610976D 00	0.14934212030D 00	0.62248285538D-01
	0.56620513932D 00	0.31735300457D 00	0.38144604158D 00	0.21907613955D 00	0.95160448859D-01
		0.34193339673D 00	0.41743338752D 00	0.26921842477D 00	0.12462759734D 00
				0.29548020197D 00	0.14959589745D 00
					0.16915580309D 00
					0.18260288610D 00
					0.18944997850D 00

α	GAUSS 5 PT	WEDDLE	GAUSS 7 PT	GAUSS 10 PT	GAUSS 16 PT
1.0200	0.23911061285D 00	0.11037834510D 00	0.12988953784D 00	0.66706265346D-01	0.27152834408D-01
	0.47691138135D 00	0.41277163589D 00	0.27944459937D 00	0.14942540291D 00	0.62253048760D-01
	0.56795601160D 00	0.29839654923D 00	0.38176493951D 00	0.21909114759D 00	0.95158804266D-01
		0.35690693958D 00	0.41780184655D 00	0.26925796001D 00	0.12462879122D 00
				0.29551922414D 00	0.14959606105D 00
					0.16915645764D 00
					0.18260341734D 00
					0.18945058532D 00
1.0300	0.23808296455D 00	0.11031184136D 00	0.12964124884D 00	0.66680049885D-01	0.27152501178D-01
	0.47764692477D 00	0.42162071500D 00	0.27958715172D 00	0.14944314718D 00	0.62253449179D-01
	0.56854022136D 00	0.28109080423D 00	0.38182454439D 00	0.21908922409D 00	0.95158569420D-01
		0.37395327883D 00	0.41789411010D 00	0.26926408108D 00	0.12462893281D 00
				0.29552349777D 00	0.14959601081D 00
					0.16915650666D 00
					0.18260342134D 00
					0.18945060860D 00
1.0400	0.23759604362D 00	0.10996071595D 00	0.12955474209D 00	0.66674082636D-01	0.27152464744D-01
	0.47802004331D 00	0.42912129600D 00	0.27964529221D 00	0.14944827645D 00	0.62253509813D-01
	0.56876782614D 00	0.26547489250D 00	0.38183655106D 00	0.21908779710D 00	0.95158522970D-01
		0.39088619111D 00	0.41792682927D 00	0.26926569073D 00	0.12462896421D 00
				0.29552415308D 00	0.14959599271D 00
					0.16915651836D 00
					0.18260341618D 00
					0.18945061103D 00
1.0500	0.23733874862D 00	0.10946099472D 00	0.12951934093D 00	0.66672349634D-01	0.27152459509D-01
	0.47823008607D 00	0.43565069749D 00	0.27967235078D 00	0.14945005646D 00	0.62253520093D-01
	0.56886233061D 00	0.25138136088D 00	0.38183776507D 00	0.219087085540 00	0.95158516648D-01
		0.40701389383D 00	0.41794108642D 00	0.26926625367D 00	0.12462896475D 00
				0.29552425469D 00	0.14959599677D 00
					0.16915651229D 00
					0.18260342099D 00
					0.18945060895D 00
1.1000	0.23698875159D 00	0.10662064762D 00	0.12948726779D 00	0.66671363049D-01	0.27156825570D-01
	0.47855235395D 00	0.45915100052D 00	0.27970214514D 00	0.14945130513D 00	0.62240153857D-01
	0.56891778892D 00	0.19810518348D 00	0.38183214181D 00	0.21908640084D 00	0.95179201229D-01
		0.47224633677D 00	0.41795689052D 00	0.26926669746D 00	0.12460545889D 00
				0.29552423353D 00	0.14961776921D 00
					0.16913957513D 00
					[illegible]

α	GAUSS 5 PT	WEDDLE	GAUSS 7 PT	GAUSS 10 PT
1.1500	0.236941936610 00	0.104378651390 00	0.129485289440 00	0.666713426730-01
	0.478606625660 00	0.473741139310 00	0.279704812600 00	0.149451345110 00
	0.568902875470 00	0.163636697400 00	0.381830565160 00	0.219086367800 00
		0.516487023810 00	0.417958665610 00	0.269266717070 00
				0.295524227350 00
1.2000	0.236931644610 00	0.102786081780 00	0.129485033090 00	0.666713605760-01
	0.478620747010 00	0.483484677130 00	0.279705250180 00	0.149451298300 00
	0.568895216750 00	0.140182533170 00	0.381830199830 00	0.219086430170 00
		0.547093415840 00	0.417959033800 00	0.269266664150 00
				0.295524246800 00
1.2500	0.236928670420 00	0.101649125660 00	0.129484984890 00	0.666713413020-01
	0.478625392910 00	0.490298538120 00	0.279705348720 00	0.149451368020 00
	0.568891873340 00	0.123604378490 00	0.381830100220 00	0.219086318270 00
		0.568895915450 00	0.417959132360 00	0.269266770980 00
				0.295524201430 00
1.3000	0.236927640500 00	0.100820413530 00	0.129484972310 00	0.666716983800-01
	0.478627173570 00	0.495228996760 00	0.279705376290 00	0.149450370990 00
	0.568890371870 00	0.111526992090 00	0.381830069580 00	0.219087605300 00
		0.584847195230 00	0.417959163620 00	0.269265731280 00
				0.295524594050 00
1.4000	0.236927056430 00	0.997336727730-01	0.129484960880 00	0.666701828130-01
	0.478628288450 00	0.501679588830 00	0.279705401270 00	0.149454367420 00
	0.568889310240 00	0.956308441410-01	0.381830037410 00	0.219082936710 00
		0.605911788520 00	0.417959200880 00	0.269269063680 00
				0.295523449380 00
1.5000	0.236926934860 00	0.990851109910-01	0.129484965430 00	
	0.478628551230 00	0.505532745470 00	0.279705391520 00	
	0.568889027830 00	0.860874941780-01	0.381830051390 00	
		0.618589298720 00	0.417959183320 00	
1.7500	0.236926884630 00	0.982993445180-01	0.129485184640 00	
	0.478628663910 00	0.510214309850 00	0.279704705840 00	
	0.568888902920 00	0.744471288450-01	0.381831129340 00	
		0.634078433570 00	0.417957960360 00	

a	GAUSS 5 PT	WEDDLE
2.0000	0.236926886621D 00	0.979853539230-01
	0.478628666930 00	0.512091017340 00
	0.568888893720 00	0.697675168420-01
		0.640312223780 00
2.5000	0.236926893830 00	0.976552230060-01
	0.478628634860 00	0.514063332810 00
	0.568888942620 00	0.648496324380-01
		0.646863623480 00
3.0000	0.236927115570 00	
	0.478628009430 00	
	0.568889749990 00	

Double Integral Norm - No Precision $||R_n||_{E_\rho}$

a	Trapezoid	Simpson	Weddle	Gauss 2 pt.	Gauss 3 pt.
1.0001	0.1046679688D (02)	0.1040579633D (02)	0.1015813168D (02)	0.1037516157D (02)	0.1031361905D (02)
1.007	0.3429901789D (01)	0.3238979278D (01)	0.2390048736D (01)	0.3139171732D (01)	0.2929577989D (01)
1.009	0.3193373174D (01)	0.2987372256D (01)	0.2087068701D (01)	0.2878874551D (01)	0.2649245386D (01)
1.01	0.3097949015D (01)	0.2885142865D (01)	0.1965979596D (01)	0.2772668248D (01)	0.2533829829D (01)
1.02	0.2519498468D (01)	0.2252723751D (01)	0.1263193949D (01)	0.2108123375D (01)	0.1797116171D (01)
1.03	0.2215750725D (01)	0.1907241079D (01)	0.9192148104D (00)	0.1739299541D (01)	0.1382887314D (01)
1.04	0.2013169205D (01)	0.1668661045D (01)	0.7006807725D (00)	0.1484059947D (01)	0.1102067051D (01)
1.05	0.1862441370D (01)	0.1485755627D (01)	0.5474355989D (00)	0.1290194900D (01)	0.8971839975D (00)
1.10	0.1422128172D (01)	0.9274917925D (00)	0.1938722321D (00)	0.7319181114D (00)	0.3845184443D (00)
1.15	0.1177422484D (01)	0.6244368429D (00)	0.8376580587D(-01)	0.4643284623D (00)	0.1967661475D (00)
1.20	0.1006455307D (01)	0.4376964426D (00)	0.4098791875D(-01)	0.3140193301D (00)	0.1118237376D (00)
1.25	0.8747842764D (00)	0.3164936090D (00)	0.2186564358D(-01)	0.2221128174D (00)	0.6821666267D(-01)
1.30	0.7681411570D (00)	0.2348739814D (00)	0.1243492240D(-01)	0.1625287795D (00)	0.4382165030D(-01)
1.40	0.6038908744D (00)	0.1377645492D (00)	0.4622369446D(-02)	0.9383945708D(-01)	0.2021109939D(-01)
1.50	0.4833705070D (00)	0.8628845236D(-01)	0.1964236582D(-02)	0.5831350541D(-01)	0.1036395250D(-01)
1.75	0.2934172212D (00)	0.3260704862D(-01)	0.3364594293D(-03)	0.2185952001D(-01)	0.2620858280D(-02)
2.00	0.1902953996D (00)	0.1482910137D(-01)	0.8114971859D(-04)	0.9914002145D(-02)	0.8662381058D(-03)
2.50	0.9288833227D(-01)	0.4259513370D(-02)	0.8575122636D(-05)	0.2842519888D(-02)	0.1506894686D(-03)
3.00	0.5216016949D(-01)	0.1599887948D(-02)		0.1067073837D(-02)	0.3822805818D(-04)
4.00	0.2129999629D(-01)	0.3558590379D(-03)		0.2372713151D(-03)	0.4658675058D(-05)
5.00	0.1073384434D(-01)	0.1132543969D(-03)		0.7550699292D(-04)	0.9377859578D(-06)

Double Integral Norm - No Precision $||R_n||_{E_\rho}$

a	Gauss 4 pt.	Gauss 5 pt.	Gauss 7 pt.	Gauss 10 pt.	Gauss 16 pt.
1.0001	0.1025170710D (02)	0.1018941897D (02)	0.1006368619D (02)	0.9872084947D (01)	0.9477268857D (01)
1.007	0.2705462601D (01)	0.2467159628D (01)	0.1964206359D (01)	0.1253846054D (01)	0.4030136639D (00)
1.009	0.2402097288D (01)	0.2140168175D (01)	0.1604471731D (01)	0.9194422378D (00)	0.2399861229D (00)
1.01	0.2276348806D (01)	0.2004576232D (01)	0.1459192820D (01)	0.7953982417D (00)	0.1891293295D (00)
1.02	0.1471205235D (01)	0.1156506141D (01)	0.6489334602D (00)	0.2389266859D (00)	0.2759652126D(-01)
1.03	0.1035294859D (01)	0.7362638037D (00)	0.3375471460D (00)	0.9375196131D(-01)	0.6318181950D(-02)
1.04	0.7599489748D (00)	0.4960680167D (00)	0.1931017631D (00)	0.4261066051D(-01)	0.1828762292D(-02)
1.05	0.5742857894D (00)	0.3481311000D (00)	0.1178839789D (00)	0.2130144393D(-01)	0.6151147925D(-03)
1.10	0.1850910254D (00)	0.8541217118D(-01)	0.1713212542D(-01)	0.1429118935D(-02)	0.8815188046D(-05)
1.15	0.7720246127D(-01)	0.2923163190D(-01)	0.3966428479D(-02)	0.1843077151D(-03)	0.3527426277D(-06)
1.20	0.3717272214D(-01)	0.1195647405D(-01)	0.1172086458D(-02)	0.3344780719D(-04)	
1.25	0.1965927823D(-01)	0.5487896892D(-02)	0.4053323227D(-03)	0.7564584603D(-05)	
1.30	0.1112196675D(-01)	0.2735533155D(-02)	0.1568640734D(-03)	0.2002621443D(-05)	
1.40	0.4110587269D(-02)	0.8104460513D(-03)	0.2986247270D(-04)	0.1963640295D(-06)	
1.50	0.1741600505D(-02)	0.2837324786D(-03)	0.7138061585D(-05)	0.2672552133D(-07)	
1.75	0.2973710988D(-03)	0.3271112192D(-04)	0.3751644820D(-06)		
2.00	0.7163719096D(-04)	0.5743495614D(-05)	0.3502072776D(-07)		
2.50	0.7561086117D(-05)	0.3678044402D(-06)			
3.00	0.1296249079D(-05)	0.4261136510D(-07)			
4.00	0.8657541858D(-07)	0.1707253302D(-08)			
5.00	0.1102387370D(-07)				

Double Integral Norm - Precision for Constants $||R_n^*||_{E_\rho}$

a	Trapezoid	Simpson	Weddle	Gauss 2 pt.	Gauss 3 pt.
1.0001	0.7392343280D (01)	0.3170215622D (01)	0.9665920435D (00)	0.2680270777D (01)	0.1853159091D (01)
1.007	0.2389671346D (01)	0.9079122081D (00)	0.1895250367D (00)	0.7321343165D (00)	0.4510947011D (00)
1.009	0.2218731906D (01)	0.8259220332D (00)	0.1640387694D (00)	0.6612857295D (00)	0.3985829070D (00)
1.01	0.2149585351D (01)	0.7925833479D (00)	0.1540024498D (00)	0.6325325183D (00)	0.3772896336D (00)
1.02	0.1727566892D (01)	0.5872628454D (00)	0.9675245467D(-01)	0.4567957257D (00)	0.2585450493D (00)
1.03	0.1503595501D (01)	0.4778521648D (00)	0.6964101445D(-01)	0.3648487808D (00)	0.1837050022D (00)
1.04	0.1353294983D (01)	0.4051259410D (00)	0.5310904647D(-01)	0.3048369278D (00)	0.1433319575D (00)
1.05	0.1241129681D (01)	0.3517743041D (00)	0.4187929470D(-01)	0.2615505758D (00)	0.1156001457D (00)
1.10	0.9150307803D (00)	0.2065191174D (00)	0.1638226667D(-01)	0.1478027082D (00)	0.5090701262D(-01)
1.15	0.7398177473D (00)	0.1390577740D (00)	0.7891820633D(-02)	0.9747541852D(-01)	0.2768107229D(-01)
1.20	0.6234291533D (00)	0.1002297563D (00)	0.4247784858D(-02)	0.6935459578D(-01)	0.1675471481D(-01)
1.25	0.5384611579D (00)	0.7542050765D(-01)	0.2460748142D(-02)	0.5173668200D(-01)	0.1085318186D(-01)
1.30	0.4729689482D (00)	0.5851012530D(-01)	0.1504364058D(-02)	0.3989209487D(-01)	0.7375292963D(-02)
1.40	0.3777778970D (00)	0.3756548466D(-01)	0.6323275616D(-03)	0.2541321745D(-01)	0.3767137950D(-02)
1.50	0.3115657908D (00)	0.2563768033D(-01)	0.2978317096D(-03)	0.1726516730D(-01)	0.2114979918D(-02)
1.75	0.2099310136D (00)	0.1168282634D(-01)	0.6266211574D(-04)	0.7824861207D(-02)	0.6472781196D(-03)
2.00	0.1528897962D (00)	0.6205219059D(-02)	0.1776911824D(-04)	0.4147090614D(-02)	0.2500425662D(-03)
2.50	0.9271222831D(-01)	0.2284007413D(-02)	0.2419177938D(-05)	0.1524067432D(-02)	0.5575451027D(-04)
3.00	0.6264212095D(-01)	0.1043010341D(-02)		0.6956315300D(-03)	0.1719734827D(-04)
4.00	0.3432797865D(-01)	0.3132777380D(-03)		0.2088781172D(-03)	0.2830101094D(-05)
5.00	0.2171376758D(-01)	0.1253497055D(-03)		0.8357068010D(-04)	0.7162447513D(-06)

Double Integral Norm - Precision for Constants $||R_n^*||_{E_p}$

a	Gauss 4 pt.	Gauss 5 pt.	Gauss 7 pt.	Gauss 10 pt.	Gauss 16 pt.
1.0001	0.1417661616D (01)	0.11447969[illegible]8D (01)	0.8189891780D (00)	0.5630853611D (00)	0.3302808152D (00)
1.007	0.3035984536D (00)	0.2132099215D (00)	0.1122124833D (00)	0.4638165027D(-01)	0.8936586132D(-02)
1.009	0.2615520597D (00)	0.1786649101D (00)	0.8861785844D(-01)	0.3338108771D(-01)	0.5324086815D(-02)
1.01	0.2446262596D (00)	0.1649406594D (00)	0.7958455841D(-01)	0.2872245873D(-01)	0.4200703437D(-02)
1.02	0.1452779610D (00)	0.8774903585D(-01)	0.3371101510D(-01)	0.8590607281D(-02)	0.6239841924D(-03)
1.03	0.9866281887D(-01)	0.5460461818D(-01)	0.1756703300D(-01)	0.3425165015D(-02)	0.1455559067D(-03)
1.04	0.7156070021D(-01)	0.3676538996D(-01)	0.1018160318D(-01)	0.1584653700D(-02)	0.4290378337D(-04)
1.05	0.5408682705D(-01)	0.2602069408D(-01)	0.6315408029D(-02)	0.8062902078D(-03)	0.1468868150D(-04)
1.10	0.1844101181D(-01)	0.6861281113D(-02)	0.9955122358D(-03)	0.5873720217D(-04)	0.2517636854D(-06)
1.15	0.8251567443D(-02)	0.2526090191D(-02)	0.2481193366D(-03)	0.8154221551D(-05)	
1.20	0.4245319750D(-02)	0.1104680989D(-02)	0.7839232912D(-04)	0.1582096822D(-05)	
1.25	0.2386976145D(-02)	0.5391285522D(-03)	0.2882449118D(-04)	0.3804259937D(-06)	
1.30	0.1429228604D(-02)	0.2844313295D(-03)	0.1180611414D(-04)	0.1067214119D(-06)	
1.40	0.5851762843D(-03)	0.9334986464D(-04)	0.2489668355D(-05)	0.1469659529D(-07)	
1.50	0.2714648030D(-03)	0.3578247266D(-04)	0.6515609209D(-06)		
1.75	0.5609504949D(-04)	0.4992330298D(-05)	0.4147327196D(-07)		
2.00	0.1579374804D(-04)	0.1024470128D(-05)			
2.50	0.2136702816D(-05)	0.8409211875D(-07)			
3.00	0.4453774465D(-06)	0.1187412312D(-07)			
4.00	0.4016920187D(-07)				
5.00	0.6430600002D(-08)				